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**THERMAL AND CREEP EFFECTS  
IN WORK-HARDENING ELASTIC-PLASTIC SOLIDS**

by

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THERMAL AND CREEP EFFECTS IN WORK-HARDENING ELASTIC-PLASTIC  
SOLIDS<sup>1</sup>

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Extremum principles governing the isothermal deformation of a work-hardening elastic-plastic solid have been given by Hodge and Prager<sup>2</sup> and Hill<sup>3</sup>. In the present note it is shown how these principles can be extended to include thermal and creep effects.

Using rectangular Cartesian coordinates  $x_i$  ( $i = 1, 2, 3$ ), denote the infinitesimal displacement from the standard state by  $u_i$ , the infinitesimal strain by  $\epsilon_{ij}$ , the stress by  $\sigma_{ij}$ , the mean normal stress by  $\sigma$ , and the stress deviation by  $s_{ij}$ . The mean normal stress is defined as

$$\sigma = \frac{1}{3} \sigma_{ii} , \quad (1)$$

where the usual summation convention regarding repeated subscripts is used; the stress deviation is defined as

$$s_{ij} = \sigma_{ij} - \sigma \delta_{ij} , \quad (2)$$

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<sup>1</sup>The results presented in this paper were obtained in the course of research conducted under Contract N7onr-35801 between the Office of Naval Research and Brown University.

<sup>2</sup>P. Hodge and W. Prager, A Variational Principle for Plastic Materials with Strain Hardening, J. Math. Phys. 27, 1-10(1948).

<sup>3</sup>R. Hill, The Mathematical Theory of Plasticity, Clarendon Press, Oxford, 1950, pp.63-66.

$\delta_{ij}$  being the Kronecker delta. The mean normal stress is an invariant of the stress tensor. Further invariants useful in the theory of isotropic plastic solids are

$$J_2 = \frac{1}{2} s_{ij} s_{ji}, J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki}. \quad (3)$$

To specify the mechanical behavior of the isotropic solid, consider an element that, at the generic instant  $t$ , has the temperature  $\theta$  and the strain  $\epsilon_{ij}$  and is under the stress  $\sigma_{ij}$ . In the interval between the instants  $t$  and  $t + dt$ , let the temperature be changed by  $d\theta$  and the stress by  $d\sigma_{ij} = ds_{ij} + d\sigma\delta_{ij}$ .

The corresponding change of strain,  $d\epsilon_{ij}$ , will then be assumed to consist of the following components:

1) the elastic component

$$d\epsilon_{ij}^e = \alpha(\theta) ds_{ij} + \beta(\theta) d\sigma\delta_{ij}, \quad (4)$$

where  $\alpha(\theta)$  is one half of the reciprocal of the temperature-dependent shear modulus, and  $\beta(\theta)$  is one third of the reciprocal of the temperature-dependent bulk modulus;

2) the thermal component

$$d\epsilon_{ij}^\theta = [\alpha'(\theta) s_{ij} + \beta'(\theta) \sigma\delta_{ij}] d\theta + \gamma(\theta) \delta_{ij} d\theta \quad (5)$$

where the prime denotes differentiation with respect to the temperature and  $\gamma(\theta)$  is the coefficient of linear thermal expansion at zero stress;

3) the creep component

$$d\epsilon_{ij}^c = \Phi(\theta, J_2, J_3) \frac{\partial \Phi}{\partial \sigma_{ij}} dt, \quad (6)$$

where  $\Phi = \Phi(\theta, J_2, J_3)$ ; and

4) the plastic component

$$d\epsilon_{ij}^p = \begin{cases} 0 & \text{if } \psi(\theta, J_2, J_3) < h, \\ \Phi(\theta, J_2, J_3) \frac{\partial \psi}{\partial \sigma_{ij}} (d\psi + |d\psi|) & \text{if } \psi(\theta, J_2, J_3) = h, \end{cases} \quad (7)$$

where the scalar  $h$  describes the state of hardening of the considered element at the time  $t$ .

Equations (4) and (5) result from a generalized form of Hooke's law in which the elastic constants are functions of the temperature.

In Eq.(6) the derivative  $\partial \Phi / \partial \sigma_{ij}$  must be evaluated for the state of stress existing at the time  $t$ . The expression  $\Phi \partial \Phi / \partial \sigma_{ij}$  then represents the rate of secondary creep corresponding to this state of stress. This expression is sufficiently general to comprise all isotropic laws of secondary creep that have so far been proposed in the literature on creep.

In (7), the equation  $\psi(\theta, J_2, J_3) = h$  represents the temperature-dependent yield limit for the state of hardening achieved at the instant  $t$ . The first line of (7) therefore states that there will be no change in plastic strain where the state of stress is below the yield limit. The differential  $d\psi$  in the second line of (7) must be evaluated from the given temperature and stress at the instant  $t$  and the changes in temperature and

stress during the considered time interval. On account of the absolute value in the second line of (7), there will be no change in plastic strain even if the state of stress is at the yield limit provided that  $d\psi < 0$ .

The extremum principles that are to be established concern the following boundary value problem. Consider a mass of work-hardening plastic material that has been deformed and, at the time  $t$ , occupies a region  $V$  bounded by the surface  $S$ . Suppose that the temperature  $\theta$ , the stress  $\sigma_{ij}$ , and the state of hardening  $h$  are known throughout  $V$ . If the unit vector along the exterior normal of  $S$  is denoted by  $n_j$ , the surface traction in the considered state is  $T_i = \sigma_{ij}n_j$ . Prescribe now infinitesimal changes  $d\theta$  of the temperature throughout  $V$ , infinitesimal changes  $dT_i$  of the surface traction on the portion  $S_T$  of the surface and infinitesimal displacements  $dU_i$  on the remainder  $S_U$  of the surface. What are the corresponding changes of stress  $d\sigma_{ij}$  and the corresponding displacement  $du_i$  throughout  $V$ ?

The infinitesimal displacement  $du_i$  causes the strain to change by

$$d\epsilon_{ij} = \frac{1}{2} \left[ \frac{\partial}{\partial x_i}(du_j) + \frac{\partial}{\partial x_j}(du_i) \right]. \quad (8)$$

The change of stress must satisfy the equation of equilibrium which, to within higher order terms, can be written as

$$\frac{\partial}{\partial x_j}(d\sigma_{ij}) = 0. \quad (9)$$

Finally, the following boundary conditions must be satisfied

$$d\sigma_{ij} n_j = dT_i \text{ on } S_T, \quad (10)$$

$$du_i = dU_i \text{ on } S_U \quad (11)$$

The problem thus consists in determining the change of stress  $d\sigma_{ij}$  and the displacement  $du_i$  in such a manner that the boundary conditions (10) and (11) and the equation of equilibrium(9) are satisfied, and that the strain change computed from (8) is related to the given values of  $\theta$ ,  $d\theta$ ,  $\sigma_{ij}$  and  $h$  and to the sought stress change  $d\sigma_{ij}$  by means of Eqs.(4) through (7).

The first extremum principle compares the actual changes of stress and strain,  $d\sigma_{ij}$  and  $d\epsilon_{ij}$ , to a fictitious change of stress  $d\sigma_{ij}^*$  and the corresponding change of strain  $d\epsilon_{ij}^*$ . The stress change  $d\sigma_{ij}^*$  is supposed to satisfy the equation of equilibrium and the boundary condition on  $S_T$ , and the strain change  $d\epsilon_{ij}^*$  is associated with  $\theta$ ,  $d\theta$ ,  $\sigma_{ij}$ ,  $h$ , and  $d\sigma_{ij}^*$  by means of Eqs.(4) through (7), but need not be derivable from a displacement field. The principle of virtual work then furnishes the equation

$$\int [(d\sigma_{ij}^* - d\sigma_{ij}) d\epsilon_{ij}] dV = \int [(dT_i^* - dT_i) dU_i] dS_U, \quad (12)$$

where  $dT_i^* = d\sigma_{ij}^* n_j$ .

The integrand of the left-hand side of (12) can be transformed as follows:

$$2(d\sigma_{ij}^* - d\sigma_{ij})d\epsilon_{ij} = (d\sigma_{ij}^* d\epsilon_{ij}^* - d\sigma_{ij} d\epsilon_{ij}) - [d\sigma_{ij}^*(d\epsilon_{ij}^* - d\epsilon_{ij}) + d\epsilon_{ij}(d\sigma_{ij} - d\sigma_{ij}^*)] . \quad (13)$$

On the right-hand side of this equation, introduce the components (4) through (7) of the change of strain. Since  $d\epsilon_{ij}^{*\theta} = d\epsilon_{ij}^\theta$  and  $d\epsilon_{ij}^{*c} = d\epsilon_{ij}^c$ ,

$$2(d\sigma_{ij}^* - d\sigma_{ij}) d\epsilon_{ij} = d\sigma_{ij}^*(d\epsilon_{ij}^* + d\epsilon_{ij}^{*\theta} + d\epsilon_{ij}^{*c}) - d\sigma_{ij}(d\epsilon_{ij} + d\epsilon_{ij}^\theta + d\epsilon_{ij}^c) - [d\sigma_{ij}^*(d\epsilon_{ij}^{*e} + d\epsilon_{ij}^{*p} - d\epsilon_{ij}^e - d\epsilon_{ij}^p) + (d\epsilon_{ij}^e + d\epsilon_{ij}^p)(d\sigma_{ij} - d\sigma_{ij}^*)] . \quad (14)$$

The bracket in (14) involves only elastic and plastic changes of strain. When the case  $d\sigma_{ij}^* \equiv d\sigma_{ij}$  is excluded, it can be shown that this bracket is positive. This is done in exactly the same manner as in the proof of the extremum principle for isothermal deformation of a work-hardening plastic material (see Hill<sup>3</sup>, pp.63-64). Thus, the left-hand side of (14) is smaller than the first two terms on the right-hand side unless  $d\sigma_{ij}^* \equiv d\sigma_{ij}$ . When this result is introduced into (12), the following relation is obtained:

$$\frac{1}{2} \int [d\sigma_{ij}^*(d\epsilon_{ij}^* + d\epsilon_{ij}^{*\theta} + d\epsilon_{ij}^{*c})] dV - \int (dT_1^* dU_1) dS_U \geq \frac{1}{2} \int [d\sigma_{ij}(d\epsilon_{ij} + d\epsilon_{ij}^\theta + d\epsilon_{ij}^c)] dV - \int (dT_1 dU_1) dS_U , \quad (15)$$

where the equality sign holds only if  $d\sigma_{ij}^* \equiv d\sigma_{ij}$ . The relation (14) establishes a minimum property of the actual change of state.

Before the second extremum principle can be discussed, it must be shown that Eqs.(4) through (7) can be transformed so as to represent  $d\sigma_{ij}$  as function of  $d\epsilon_{ij}$ . We note first that  $d\epsilon_{ij}^{\theta}$  and  $d\epsilon_{ij}^c$  follow immediately from the data of the considered boundary value problem. Since these two components of the change of strain are known, giving  $d\epsilon_{ij}$  is equivalent to giving  $d\epsilon_{ij}^c + d\epsilon_{ij}^p = d\epsilon_{ij} - d\epsilon_{ij}^{\theta} - d\epsilon_{ij}^c$ . The proof that the sum  $d\epsilon_{ij}^c + d\epsilon_{ij}^p$  specifies a unique  $d\sigma_{ij}$  then proceeds exactly as in the case where  $d\epsilon_{ij}^c + d\epsilon_{ij}^p$  represents the entire change of strain (see Hill<sup>3</sup>, pp.68-69).

The second extremum principle compares the actual changes of strain and stress,  $d\epsilon_{ij}$  and  $d\sigma_{ij}$ , to a fictitious change of strain  $d\epsilon_{ij}^*$  and the corresponding change of stress  $d\sigma_{ij}^*$ . The strain change  $d\epsilon_{ij}^*$  is supposed to be derivable from a displacement field  $du_i^*$  that satisfies the boundary conditions on  $S_U$ ; the stress change  $d\sigma_{ij}^*$  is associated with  $\theta$ ,  $d\theta$ ,  $\sigma_{ij}$ ,  $h$ , and  $d\epsilon_{ij}^*$  by means of Eq.(4) through (7), but need not satisfy the equation of equilibrium or the boundary condition on  $S_T$ . The principle of virtual work then furnishes the equation

$$\int [d\sigma_{ij}(d\epsilon_{ij}^* - d\epsilon_{ij})] dV = \int [dT_i(du_i^* - du_i)] dS_T. \quad (16)$$

The integrand on the left-hand side of (16) equals

$$d\sigma_{ij} (d\epsilon_{ij}^{*e} + d\epsilon_{ij}^{*p} - d\epsilon_{ij}^e - d\epsilon_{ij}^p) \quad (17)$$



and does not involve the thermal or creep effects; it can therefore be transformed in exactly the same manner as in the case where the sum of elastic and plastic strain changes represents the total strain change (see Hill<sup>3</sup>, pp.65-66). As a result of of this transformation, the following relation is obtained:

$$\int (dT_i du_i^*) dS_T - \frac{1}{2} \int [d\sigma_{ij}^* (d\epsilon_{ij}^{*e} + d\epsilon_{ij}^{*p})] dV$$

$$\leq \int (dT_i du_i) dS_T - \frac{1}{2} \int [d\sigma_{ij} (d\epsilon_{ij}^e + d\epsilon_{ij}^p)] dV, \quad (18)$$

where the equality sign holds only if  $d\epsilon_{ij}^* = d\epsilon_{ij}$ . The relation (18) establishes a maximum property of the actual change of state.